The Riemann-Hilbert Problem and Integrable Systems

Alexander R. Its

In its original setting, the Riemann-Hilbert problem is the question of surjectivity of the monodromy map in the theory of Fuchsian systems.

An $N \times N$ linear system of differential equations

$$\frac{d\Psi(\lambda)}{d\lambda} = A(\lambda)\Psi(\lambda) \tag{1}$$

is called *Fuchsian* if the $N \times N$ coefficient matrix $A(\lambda)$ is a rational function of λ whose singularities are simple poles. Each Fuchsian system generates, via analytic continuation of its fundamental solution $\Psi(\lambda)$ along closed curves, a representation of the fundamental group of the punctured Riemann sphere (punctured at the poles of $A(\lambda)$) in the group of $N \times N$ invertible matrices. This representation (or rather its conjugacy class) is called the *monodromy group* of equation (1), and it is the principal object of the theory of Fuchsian systems. The question of *whether there always exists a Fuchsian system with given poles and a given monodromy group* was included by Hilbert in his famous list as problem number twenty-one. The problem got the name "Riemann-Hilbert" for its obvious relation to the general idea of Riemann that an analytic (vector-valued) function could be completely defined by its singularities and monodromy properties.

Subsequent developments put the Riemann-Hilbert problem into the context of analytic factorization of matrix-valued functions and brought to the area the methods of singular integral equations (Plemelj, 1908) and holomorphic vector bundles (Röhrl, 1957). This resulted eventually in a negative (!) solution, due to Bolibruch (1989), of the Riemann-Hilbert problem in its original setting and to a number of deep results (Bolibruch, Kostov) concerning a thorough analysis of relevant solvability conditions. We refer the reader to the book of Anosov and Bolibruch [2] for more on Hilbert's twenty-first problem and the fascinating history of its solution (and for more details on the genesis of the name "Riemann-Hilbert").

Simultaneously, and to a great extent independently of the solution of the Riemann-Hilbert problem itself, a powerful analytic apparatus—the Riemann-Hilbert method—was developed for solving a vast variety of problems in pure and applied mathematics. The Riemann-Hilbert method reduces a particular problem to the reconstruction of an analytic function from jump conditions or, equivalently, to the analytic factorization of a given matrix- or scalar-valued function defined on a curve. Following a tradition that developed in mathematical physics, it is these problems, and not just the original Fuchsian one, that we will call *Riemann-Hilbert problems.*[1] In other words, we are adopting a point of view according to which *the* Riemann-Hilbert (monodromy) problem is formally treated as a special case (although an extremely important one) of *a* Riemann-Hilbert (factorization) problem. The latter is viewed as an analytic tool, but one whose implementation is not at all algorithmic and which might use quite sophisticated and

---

Alexander R. Its is professor of mathematics at Indiana University–Purdue University at Indianapolis. His email address is `itsa@math.iupui.edu`.

The author would like to thank Harold Boas for many excellent comments which helped the author organize his ideas about the topic of the paper. This work was supported in part by NSF Grant DMS-0099812.

[1] *It should be mentioned that in the theory of boundary values of analytic functions the problem of reconstructing a function from its jumps across a curve is sometimes called the "Hilbert boundary-value problem". This adds even more subtlety to the origin of the name "Riemann-Hilbert problem".*

"custom-made" analytic ideas (depending on the particular setting of the factorization problem).

A classical example of the use of analytic factorization techniques is the Wiener-Hopf method in linear elasticity, hydrodynamics, and diffraction.

The goal of this article is to present some new developments in the Riemann-Hilbert formalism which go far beyond the classical Wiener-Hopf schemes and, at the same time, have many important similarities with the analysis of the original Fuchsian Riemann-Hilbert problem. These developments come from the theory of *integrable systems*.

The modern theory of integrable systems has its origin in the seminal works of Gardner, Green, Kruskal, and Miura (1967), Lax (1968), Faddeev and Zakharov (1971), and Shabat and Zakharov (1971), where what is now widely known as the Inverse Scattering Transform (IST) method in soliton theory originated (we will recall the essence of the IST method later on; we point out the monograph [16] as the principal reference). Integrable systems is currently an expanding area that includes the analysis of exactly solvable quantum field and statistical physics models; the theory of integrable nonlinear partial differential equations (PDEs) and ordinary differential equations (ODEs)—equations of KdV and Painlevé types; and quantum and classical dynamical systems that are integrable in the sense of Liouville. During the last thirty years, the theory of integrable systems has developed into an important part of mathematical physics and analysis, and it has become one of the principal sources of new analytic and algebraic ideas for many branches of contemporary mathematics and theoretical physics.[2] The most recent "beneficiaries" are orthogonal polynomials, combinatorics, and random matrices.

We call a system of (nonlinear) differential equations an *integrable system* if it can be represented as a compatibility condition of an auxiliary overdetermined *linear* system of differential equations. Following the tradition in soliton theory, we call this auxiliary linear system a *Lax pair*[3] of the given (nonlinear) system, even though it actually might involve more than two equations. We also require that the Lax pair depend rationally on an auxiliary complex parameter, which is called a *spectral parameter*. This requirement is crucial:[4] it makes an integrable system *completely* integrable in the sense of Liouville and, even more importantly, makes possible an effective evaluation of the commuting integrals of motion, the invariant submanifolds, and the corresponding angle variables (that is, an effective realization of the Liouville-Arnold integration algorithm). Indeed, the presence of the spectral parameter in the Lax pair brings the tools of complex analysis into the problem, and this ultimately transforms the original problem of solving a system of differential equations into the question of reconstructing an analytic function from the known structure of its singularities. In turn, this question (almost) always can be reformulated as a Riemann-Hilbert problem of finding an analytic function (generally matrix-valued) from a prescribed jump condition across a curve. The Riemann-Hilbert problem, especially in the matrix case, might itself still be a transcendental one. But even then it describes the solution of the differential system in terms *independent of the theory of differential equations*. In this sense, the original differential system is "solved". In fact, the solution might even be explicit: namely, given in terms of elementary or elliptic or abelian functions and a finite number of contour integrals of such functions. In general, the Riemann-Hilbert formalism provides a representation in terms of the solutions of certain linear singular integral equations, which in turn can be related to the theory of infinite-dimensional Grassmannians and holomorphic vector bundles.

This notion of integrable systems and the Riemann-Hilbert method of solving them was essentially worked out in the 1970s and 1980s in the theory of nonlinear PDEs of KdV type, that is, in the soliton theory. Since then, the Riemann-Hilbert approach has gradually become a quite universal analytic tool for studying problems from many areas of modern mathematics not previously considered as "integrable systems". Moreover, some of these problems in their initial setting are not necessarily differential systems at all.

In this article we will describe the application of the Riemann-Hilbert formalism to integrable systems, emphasizing the analytic aspects. We shall start by explaining in more detail what Riemann-Hilbert (factorization) problems are and what the advantage is of reducing a problem to Riemann-Hilbert type. Then we will consider the

[2] *Perhaps the most celebrated example of such influence is quantum groups, which emerged out of the works of Faddeev, Sklyanin, Takhtajan, and other members of the Leningrad group on the quantum version of the IST. A more recent example is the quite remarkable appearance, in the Seiberg-Witten $N = 2$ supersymmetric gauge theory, of the so-called algebraically integrable systems (the Liouville tori are Jacobi varieties), which have their roots in the theory of periodic solutions of integrable PDEs developed in the 1970s (see the review paper [9] and the monographs [16] and [4] for the history and the main references concerning the periodic version of the IST).*

[3] *Strictly speaking, the compatibility-condition generalization of the original Lax-equation formalism came after Lax's paper; it first appeared in 1974, in the work of Novikov (periodic problem for KdV) and of Ablowitz, Kaup, Newell, and Segur (sine-Gordon equation).*

[4] *In fact, the more general settings due to Hitchin (1987) and Krichever (2001) allow the spectral parameter to vary in an algebraic curve. Also, one can consider difference or differential-difference Lax pairs as well.*

appearance and use of Riemann-Hilbert problems in the theory of special functions of Painlevé type. Simultaneously, we will see that this area indeed falls into the category of integrable systems.

Choosing Painlevé functions (all the necessary definitions and historical references will be given later) as a principal example enables us to introduce the Riemann-Hilbert scheme in a rather elementary, although sufficiently general, manner. Also, this will put us in the context of the Fuchsian monodromy theory where both the "original" Riemann-Hilbert problem and the Painlevé equations belong. The drawback of the approach is that we will not have room for many other exciting applications, which range from integrable PDEs of KdV type to exactly solvable quantum field and statistical mechanics models and (most recently) to the theory of orthogonal polynomials, matrix models, and random permutations. It is, however, worth mentioning that in all these areas the Painlevé functions play an important role as the relevant "nonlinear special functions".

Here are some key references where the interested reader can find material concerning the topics mentioned above and which we are unable to cover in this article. The Riemann-Hilbert method for integrable PDEs originated in the works of Manakov, Shabat, and Zakharov done in 1975–1979, and since then it has been widely used in soliton theory. We refer the reader to the monographs [16], [10], [1], and [3] for a detailed presentation of the different aspects of the method. The Riemann-Hilbert approach to quantum exactly solvable models was initiated in the beginning of the 1980s by Jimbo, Miwa, Môri, and Sato, and it was further developed in the late 1980s and in the 1990s in the series of works of Izergin, Korepin, Slavnov, Deift, Zhou, and this author. The method is presented in the monograph [14] (see also the more recent survey [5]). The Riemann-Hilbert approach to orthogonal polynomials and matrix models was suggested in 1991 by Fokas, Kitaev, and this author, and recently it helped in solving some of the long-standing problems in the asymptotics of orthogonal polynomials related to universalities in random matrices (the works of Bleher, Deift, Kricherbauer, McLaughlin, Venakides, Zhou, and this author, done in the late 1990s) and random permutations (the 1999 work of Baik, Deift, and Johansson followed by an explosion of activity in the area). We refer the reader to the monograph [6] and the survey [7] for a detailed presentation of the approach and for more on its history.

Riemann-Hilbert Problems

An analytic function is uniquely determined by its singularities, in virtue of Liouville's theorem. In a way, this is the most general example of "integrability": the *local* properties of an object yield complete information about its *global* behavior.

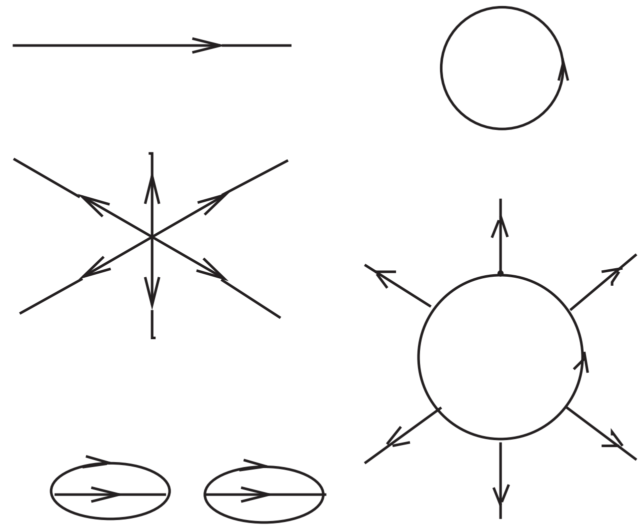

Figure 1. Typical contours Γ.

Therefore, one can suggest the most general, and hence quite tautological, "definition" of an integrable system as *a problem whose solution can be reduced to reconstructing an analytic function from the known structure of its singularities.* In turn, as indicated in the introduction, this question (almost) always can be formulated as a *Riemann-Hilbert problem*.

Roughly speaking, as already indicated, a Riemann-Hilbert problem is the problem of finding an analytic function in the complex plane having a prescribed jump across a curve.[5] More precisely, suppose Γ is an oriented contour in the complex λ-plane. The contour Γ might have points of self-intersection, and it might have more than one connected component. Figure 1 depicts typical contours appearing in applications. The orientation defines the $+$ and the $-$ sides of Γ in the usual way. Suppose in addition that we are given a map G from Γ into the set of $N \times N$ invertible matrices. The *Riemann-Hilbert problem* determined by the pair (Γ, G) consists in finding an $N \times N$ matrix-valued function $Y(\lambda)$ with the following properties.[6]

- $Y(\lambda)$ is analytic for λ in $\mathbb{C} \setminus \Gamma$.
- The limit $Y_-(\lambda)$ of Y from the minus side of Γ and the limit $Y_+(\lambda)$ from the plus side of Γ are related for $\lambda \in \Gamma$ by the equation

$$Y_+(\lambda) = Y_-(\lambda)G(\lambda).$$

5 *The more general settings are the so-called nonlocal Riemann-Hilbert problem and the $\bar\partial$-problem which were brought to the theory of integrable systems in the 1980s in the works of Ablowitz, BarYaacov, Fokas, Manakov, and Zakharov devoted to the 2+1 integrable PDEs.*

6 *It is an instructive exercise to reformulate as a Riemann-Hilbert problem the standard question of reconstructing a rational function from its poles and the principal parts at the poles.*

- $Y(\lambda)$ tends to the identity matrix I as $\lambda \to \infty$.

The precise sense in which the limit at ∞ and the limits from the two sides of Γ exist are technical matters that should be specified for each given pair (Γ, G). The highly nontrivial questions concerning the minimal restrictions on the contour Γ and the allowable functional classes for the map G are issues of the general theory of analytic matrix factorization. For a detailed exposition of this extremely interesting and deep area of modern complex analysis we refer the reader to the 1981 monograph of Clancey and Gohberg and to the 1987 monograph of Litvinchuk and Spitkovskii (see also [3], the works of Zhou on the Riemann-Hilbert approach to inverse scattering, and the most recent works of Deift and Zhou on the L_2-Riemann-Hilbert theory). The general facts established in this area, especially the ones concerning the properties of the Cauchy operators defined on contours with self-intersections, are extremely important; they provide the Riemann-Hilbert formalism with the necessary mathematical rigor.

Why should it help if a problem can be reduced to a Riemann-Hilbert problem? The advantage is immediate in the scalar case, $N = 1$. Indeed, in this case the original multiplicative jump condition can be rewritten in the additive form

$$\log Y_+(\lambda) = \log Y_-(\lambda) + \log G(\lambda).$$

An additive jump relation of the form $y_+(\lambda) = y_-(\lambda) + g(\lambda)$ can always be resolved by means of the contour integral

$$y(\lambda) = \frac{1}{2\pi i}\int_\Gamma \frac{g(\mu)}{\mu - \lambda}\,d\mu$$

(the Cauchy-Plemelj-Sokhotskii formula). In the scalar case, therefore, or more generally in the *abelian* case when

$$[G(\lambda_1), G(\lambda_2)] \equiv G(\lambda_1)G(\lambda_2) - G(\lambda_2)G(\lambda_1) = 0$$

for all λ_1 and λ_2 in Γ, the solution of the Riemann-Hilbert problem admits an explicit integral representation

$$Y(\lambda) = \exp\left\{\frac{1}{2\pi i}\int_\Gamma \frac{\log G(\mu)}{\mu - \lambda}\,d\mu\right\}. \tag{2}$$

There is a subtle matter of how to treat this equation if the problem has a nonzero index, that is, if $\partial\Gamma = 0$ and $\Delta \log G|_\Gamma \neq 0$. Still, formula (2), after a suitable modification in the case of nonzero index (see, e.g., Gakhov's monograph *Boundary Problems*), yields a contour-integral representation for the solution of the original problem. Moreover, in typical concrete situations one can evaluate the integral in (2) in closed form or, equivalently, one can find an explicit formula involving known elementary or special functions directly by examining the jump function $G(\lambda)$.[7]

In the general *nonabelian* matrix case, formula (2) does not work. A generic matrix Riemann-Hilbert problem cannot be solved explicitly (this is a common belief, not a theorem!) in terms of contour integrals. It can, however, always be reduced to the analysis of a linear singular-integral equation. Therefore replacing the original problem by a Riemann-Hilbert problem is still advantageous. Indeed, nonabelian Riemann-Hilbert problems usually arise when the original problem is *nonlinear*, so a Riemann-Hilbert reformulation effectively linearizes an originally nonlinear system.

The main benefit of reducing originally nonlinear problems to the analytic factorization of given matrix functions arises in asymptotic analysis. In typical applications, the jump matrices $G(\lambda)$ are characterized by oscillatory dependence on external large parameters, say space x and time t. The asymptotic evaluation of the solution $Y(\lambda, x, t)$ of the Riemann-Hilbert problem as $x, t \to \infty$ turns out to be in some (not all!) ways quite similar to the asymptotic evaluation of oscillatory contour integrals via the classical method of steepest descent. Indeed, after about twenty years of significant efforts by several authors, starting from the 1973 works of Shabat, Manakov, and Ablowitz and Newell (see [8] for a detailed historical review), the development of the relevant scheme of asymptotic analysis of integrable systems finally culminated in the *nonlinear steepest descent method* for oscillatory Riemann-Hilbert problems, which was introduced in 1992 by Deift and Zhou. In complete analogy to the classical method, it examines the analytic structure of $G(\lambda)$ in order to deform the contour Γ to contours where the oscillatory factors involved become exponentially small as $x, t \to \infty$, and hence the original Riemann-Hilbert problem reduces to a collection of local Riemann-Hilbert problems associated with the relevant saddle points. The noncommutativity of the matrix setting requires, however, developing several totally new and rather sophisticated technical ideas, which, in particular, enable an explicit solution of the local Riemann-Hilbert problems.[8] For more details we refer the reader to the original papers of Deift and Zhou, and also to the review article [8]. Remarkably, the final result of the analysis is as efficient as the asymptotic evaluation of the oscillatory integrals.

[7] *The possibility of an explicit factorization might actually occur in (very) special matrix cases as well; in fact, certain problems in diffraction are solved by using such factorizations.*

[8] *It is worth mentioning that as a by-product a new collection of matrix functions admitting an explicit analytic factorization has been obtained.*

Later we will illustrate this statement by an example from the modern theory of integrable ODEs.

In summary, our major point is that *to have a solution to a (nonlinear) problem represented in terms of the function $Y(\lambda)$ defined via the factorization of a given matrix function is just as good as to have the solution written in terms of contour integrals.* In other words, the Riemann-Hilbert representation extends the notion of "integral representation" to the nonlinear, noncommutative case.

We conclude this section with a few additional general remarks concerning Riemann-Hilbert problems.

(i) The following simple observation strengthens the idea of viewing the Riemann-Hilbert formalism as a noncommutative analog of contour integral representation. Let

$$L = \int_\Gamma g(\lambda)\, d\lambda \tag{3}$$

be a contour integral, and define the matrix function

$$Y(\lambda) = \begin{pmatrix} 1 & \displaystyle\int_\Gamma \frac{g(\mu)}{\mu - \lambda}\, d\mu \\ 0 & 1 \end{pmatrix}. \tag{4}$$

Assuming that all the integrals and limits make sense (e.g., $g(\lambda)$ and Γ are continuous, and Γ is bounded), we can write

$$L = -\lim_{\lambda \to \infty} [\lambda Y_{12}(\lambda)] \quad \Big(= \mathrm{res}_{\lambda=\infty}\ Y_{12}(\lambda)\Big). \tag{5}$$

On the other hand, the matrix function $Y(\lambda)$ can alternatively be defined (again by the Cauchy-Plemelj-Sokhotskii formula) as the unique solution of the Riemann-Hilbert problem determined by the pair (Γ, G), where

$$G(\lambda) = \begin{pmatrix} 1 & 2\pi i g(\lambda) \\ 0 & 1 \end{pmatrix}. \tag{6}$$

Hence the evaluation of the contour integral (3) is equivalent to the analytic factorization of the matrix function (6).[9] Since the matrices $G(\lambda)$ for different values of λ commute with each other, the equation (4) is just the integral representation (2) in the triangular case.

(ii) Let Γ be a closed Jordan curve that divides the λ-plane into two open connected sets: the interior domain Ω_+ and the exterior domain Ω_-. Let G be a constant map, say $G(\lambda) \equiv G_0$. Then the Riemann-Hilbert problem can be solved immediately: namely,

$$Y(\lambda) = \begin{cases} I & \text{for } \lambda \in \Omega_-, \\ G_0^{-1} & \text{for } \lambda \in \Omega_+. \end{cases}$$

[9] *The reader might find it amusing to try to evaluate, via the factorization of the relevant triangular matrices, standard integrals of the form $\int_a^b R(x)\,dx$ and $\int_{-\infty}^{+\infty} R(x)e^{ix}\,dx$, where $R(x)$ is a rational function.*

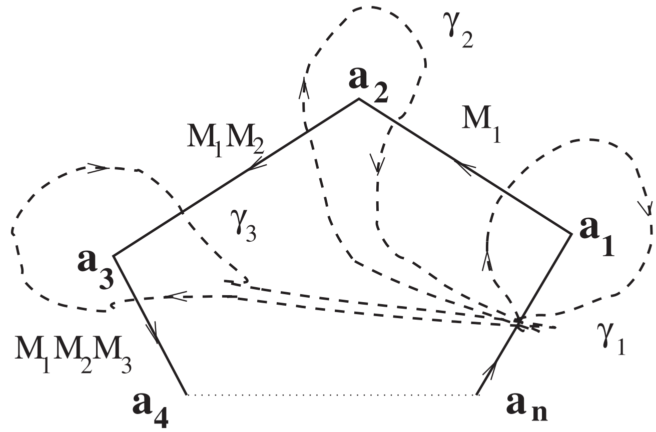

Figure 2. Fuchsian Riemann-Hilbert problem.

The easiest way to make the problem nontrivial is to let the constant map G (the jump matrix) become *piecewise constant.* The Riemann-Hilbert problem that arises in this way is exactly the kind of factorization problem that appeared in the classical work of Plemelj devoted to solving Hilbert's twenty-first problem. Figure 2 depicts the Fuchsian (i.e., piecewise-constant) Riemann-Hilbert problem in more detail. There, the contour Γ is a polygonal path, $\Gamma = [a_1, a_2] \cup [a_2, a_3] \cup \ \dots\ \cup [a_{n-1}, a_n] \cup [a_n, a_1]$, and the jump matrix $G(\lambda)$ is defined by

$$G(\lambda) = M_1 M_2 \cdots M_k, \quad \lambda \in (a_k, a_{k+1}),$$

where $\{M_1, M_2, \dots, M_n\}$ is a given set of nonsingular constant matrices. For generic M_k satisfying the cyclic relation $M_1 M_2 \cdots M_n = I$, the unique solution $Y(\lambda)$ of this Riemann-Hilbert problem exists (Plemelj). Moreover, it satisfies a Fuchsian differential equation whose poles are a_k and whose *monodromy group* is generated (see Figure 2) by the matrices $M_1, \dots, M_n$; that is,

$$\tau_{\gamma_k}(Y)(\lambda) = Y(\lambda) M_k,$$

where τ_γ denotes the operator of analytic continuation along the loop γ. This relates the Riemann-Hilbert problems with piecewise-constant jump matrices to the theory of Fuchsian systems. Plemelj used this relation in his near solution of Hilbert's twenty-first problem. A principal difficulty arises when we drop the word "generic" in the description of the given matrices M_k. In the *general* case, as was shown by Kohn and by Arnold and Il'yashenko, Plemelj's proof has gaps. The very surprising fact that these gaps cannot be closed was shown by Bolibruch by a counterexample (see [2] for further details).

(iii) As already indicated in the introduction, the modern theory of integrable systems began with the discovery of the Inverse Scattering Transform method. The essence of the method is a linearization of a nonlinear (integrable) PDE via a direct scattering transform generated by the spatial part of the corresponding Lax pair. This reduces the solution of an integrable nonlinear PDE to the solution of the inverse scattering problem for a relevant linear differential operator. It was apparently first realized by Shabat in 1979 (although some of the basic ideas can be found in old works of Krein) that the inverse scattering problem can be reformulated as a Riemann-Hilbert problem of analytic factorization of the scattering matrix. This was how the Riemann-Hilbert approach in integrable PDEs started. It is significant to notice that the "first" Riemann-Hilbert problem, i.e., the Fuchsian one, and the inverse scattering Riemann-Hilbert problem represent the two opposite ends of the whole spectrum of possible Riemann-Hilbert problems: the Fuchsian problem is the first nontrivial problem, while the inverse scattering problem is the most general one, as its jump matrix $G(\lambda)$ allows a virtually arbitrary dependence on λ.

(iv) The use of the Riemann-Hilbert problem as an analytic apparatus goes back to the beginning of the twentieth century. The main examples are the Wiener-Hopf method in linear diffraction and the theory of Toeplitz operators. The principal player in these fields is a scalar, that is, an abelian Riemann-Hilbert factorization, and the principal objects are linear PDEs. The basic reference for the classical aspects of the theory and applications of the Riemann-Hilbert problem is the 1968 monograph of Muskhelishvili. There is a very interesting revival of the "linear theme" in the recent works of Fokas, where a unified approach is suggested, based on the Riemann-Hilbert method, for solving initial-boundary-value problems for linear PDEs with constant coefficients and integrable nonlinear PDEs.

(v) Although this article deals with only the analytic aspects of the Riemann-Hilbert method, the method has two other very important and intertwining components. One is geometric: the relation to holomorphic vector bundles. Another is algebraic: the relation to loop groups. For these, see the survey [17] and the monographs [2] and [10].

"From Gauss to Painlevé"

This is the main section of the article. The title, but not the content, is borrowed from the 1991 book on special functions of Iwasaki, Kimura, Shimomura, and Yoshida.

Using the Airy equation and its natural nonlinear analog—the second Painlevé equation—as the basic examples of linear (Gauss) and nonlinear (Painlevé) special functions, we will describe the principal analytic ideas and the kind of results that can be obtained via the Riemann-Hilbert method in the theory of integrable systems. At the same time, we will explain why the special functions make integrable systems.

The Airy functions are defined as solutions of the linear ordinary differential equation

$$u_{xx} = xu. \tag{7}$$

As mentioned above, the Airy functions belong to the family of classical *special functions*. Before we proceed with the derivation of the Riemann-Hilbert formalism for the Airy function, we address the following basic issue: What is so "special" about the special functions in general and the Airy function in particular? One way to answer this question is the following.

Equation (7) is a particular case of a linear differential equation of the form

$$u_{xx} = p(x)u, \tag{8}$$

where $p(x)$ is a polynomial. Unless $p(x)$ is of a very special type, this equation cannot be solved in terms of elementary functions. However, one can always find the elementary asymptotic solutions, $u_\pm(x|\mathbf{a}_\pm)$, to the equation as $x \to \pm\infty$. Here the $\mathbf{a}_\pm$ indicate sets of asymptotic parameters. The principal question is this: Suppose that $u_+(x|\mathbf{a}_+)$ and $u_-(x|\mathbf{a}_-)$ represent the asymptotics of the same solution. Can one describe the map $\mathbf{a}_+ \mapsto \mathbf{a}_-$ in terms of elementary functions or finitely many contour integrals of elementary functions (i.e., avoiding the necessity of solving an integral equation)? In other words, does equation (8) admit *explicit connection formulae*? For a generic polynomial $p(x)$ the answer is "no",[10] but for the Airy equation the answer is "yes". From the analytic point of view, this fact justifies the title "special" for the Airy function. Similarly, the other classical special functions of hypergeometric type, such as the Bessel functions and the Whittaker functions, all are defined as solutions of second-order linear differential equations possessing this extremely *special* property—each of them admits explicit asymptotic connection formulae.[11]

The analytic mechanism that yields explicit connection formulae for Airy (as well as for Bessel, Whittaker, etc.) functions is the contour integral representations which are available for all the special functions of hypergeometric type. Therefore, according to our idea of viewing contour integrals as abelian Riemann-Hilbert problems, there should be a Riemann-Hilbert representation for the Airy functions as well. In order to obtain it, we have to recall the Airy integral formulae.

Consider the collection of six rays

[10] *Strictly speaking, this is again the general belief, not a theorem.*

[11] *It is worth noticing that asymptotic connection formulae are exactly what is most frequently needed from the special functions in applications.*

(9) $\Gamma_k = \left\{\lambda : \arg\lambda = \dfrac{2k-1}{6}\pi\right\}, \quad k = 1, 2, \ldots, 6,$

oriented towards infinity, and let Γ, indicated in bold in Figure 3, be the union of the rays Γ_2, Γ_4, and Γ_6. The classical integral representation of the general solution of the Airy equation (7) can be written as

(10)

$$u(x) = \frac{i}{\pi}\left\{s_2\int_{\Gamma_2} + s_4\int_{\Gamma_4} + s_6\int_{\Gamma_6}\right\} e^{-\frac{8i}{3}\lambda^3 - 2ix\lambda}\, d\lambda,$$

where the complex parameters s_2, s_4, and s_6 satisfy a single restriction, the *cyclic relation*[12]

(11) $$s_2 + s_4 + s_6 = 0.$$

In view of equations (3)–(6), the integral representation (10) implies that

$$u(x) = 2\lim_{\lambda\to\infty}[\lambda Y_{12}(\lambda)],$$

where the matrix function $Y(\lambda) \equiv Y(\lambda, x)$ is the solution of the (abelian) 2×2 matrix Riemann-Hilbert problem for the contour Γ. The corresponding jump matrix $G(\lambda)$ is the upper-triangular oscillatory matrix function defined by

(12) $$G(\lambda) \equiv G(\lambda, x) = \begin{pmatrix} 1 & s_k e^{-\frac{8i}{3}\lambda^3 - 2ix\lambda} \\ 0 & 1 \end{pmatrix},$$

$\lambda \in \Gamma_k$, $k = 2, 4, 6$. The oscillatory factor in (12) and the choice of the contour Γ are consistent with the normalization condition at $\lambda = \infty$. Indeed, the matrix function $G(\lambda)$ rapidly approaches the identity matrix as $\lambda \to \infty$ along the contour Γ. This makes the Airy Riemann-Hilbert problem well posed.

Similar considerations, based on the relevant contour integrals, easily produce Riemann-Hilbert representations for all the other classical special functions, including even those which are not of the hypergeometric type, e.g., the Euler Γ-function (Kitaev, 2002) and the Riemann zeta-function (see the end of this article).

Although the Riemann-Hilbert treatment of the classical special functions does not seem to have a big technical advantage over integral representations, it does lead, via a natural nonabelianization procedure, to the next, more exotic, analytic objects, the *Painlevé transcendents*.

A simple way to obtain a nonabelian generalization of the Airy Riemann-Hilbert problem is to augment the contour Γ by the three complementary rays Γ_1, Γ_3, and Γ_5 (shown in Figure 3 by dashed lines), simultaneously complementing equations (12) by three more equations defining the jump matrix on the new rays (note the change of the triangularity and of the sign of the exponent):

$$G(\lambda) \equiv G(\lambda, x) = \begin{pmatrix} 1 & 0 \\ s_k e^{\frac{8i}{3}\lambda^3 + 2ix\lambda} & 1 \end{pmatrix},$$

$\lambda \in \Gamma_k$, $k = 1, 3, 5$. Along the augmented contour Γ, we still have the asymptotic consistency condition that $G(\lambda)$ tends to the identity matrix as $\lambda \to \infty$. The nonabelian extension leads to the following nonlinearization of the Airy cyclic relation (11):

(13) $$s_{k+3} = -s_k, \quad k = 1, 2, 3, \quad s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0.$$

We will see shortly that, just as in the case of their linear counterpart, these relations are needed to eliminate the possible singularity of the function $Y(\lambda)$ at $\lambda = 0$.

Let $Y(\lambda, x)$ be the solution of this nonabelian Riemann-Hilbert problem, and define the function $u(x)$ again by the same limit formula as in the linear case; i.e., put

(14) $$u(x) = 2\lim_{\lambda\to\infty}[\lambda Y_{12}(\lambda)].$$

Then, in place of the *linear* Airy equation (7), the following *nonlinear* second-order differential equation arises:

(15) $$u_{xx} = xu + 2u^3.$$

Unlike in the linear case, the proof of (15) is not straightforward. Indeed, we no longer have an explicit formula for $Y(\lambda)$ or for $u(x)$, so the very existence of the solution $Y(\lambda)$ of the Riemann-Hilbert problem and its "good" analytic properties with respect to the parameter x are now quite nontrivial analytic facts. They can be established by using

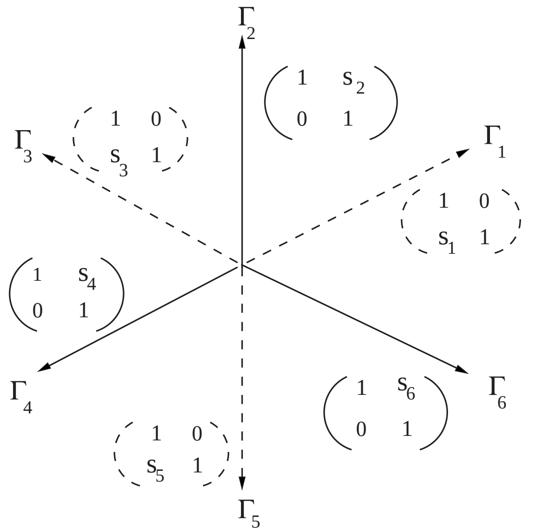

Figure 3. Airy-Painlevé Riemann-Hilbert problem.

[12] *The proof of the integral representation for the Airy functions is straightforward. Differentiating under the integral shows that $u_{xx} - xu$ is equal to a constant times $\left\{s_2\int_{\Gamma_2} + s_4\int_{\Gamma_4} + s_6\int_{\Gamma_6}\right\} d\left(e^{-\frac{8i}{3}\lambda^3 - 2ix\lambda}\right)$, and this expression equals 0 because of the cyclic relation and the asymptotic behavior $e^{-\frac{8i}{3}\lambda^3 - 2ix\lambda} \to 0$ as $\lambda \to \infty$ in Γ.*

techniques based on the Fredholm analysis of the associated singular integral equation or by applying methods of holomorphic vector bundles based on the generalized Birkhoff-Grothendieck theorem with parameters. A precise statement concerning the solution $Y(\lambda) \equiv Y(\lambda, x)$ reads as follows.[13]

Theorem 1. *Suppose the set $s \equiv (s_1, \dots, s_6)$ satisfies the cyclic relation (13). Then there exists a countable subset X_s of the complex x-plane, having the point at ∞ as its only accumulation point, and a matrix function $Y(\lambda, x)$ solving the nonabelian Airy Riemann-Hilbert problem for all $x \notin X_s$. Moreover, if Ω_k denotes the sector in the complex λ-plane bounded by the rays Γ_{k-1} and $-\Gamma_k$, then each restriction $Y_k(\lambda, x) \equiv (Y|_{\Omega_k})(\lambda, x)$ is holomorphic in $\overline{\Omega_k} \times (\mathbb{C} \setminus X_s)$ and meromorphic along $\overline{\Omega_k} \times X_s$. The normalization condition at $\lambda = \infty$ extends to the full asymptotic series*

$$Y(\lambda, x) \sim I + \sum_{j=1}^{\infty} \frac{m_j(x)}{\lambda^j}, \quad \lambda \to \infty, \tag{16}$$

which is differentiable with respect to λ and x. The coefficient functions $m_j(x)$ are meromorphic in x and have the set X_s as the set of their poles.

The x-meromorphicity is a new analytic feature of the function $Y(\lambda, x)$ in the nonabelian case (in the linear—abelian—case, the function $Y(\lambda, x)$ is entire with respect to x). In fact, the solutions of all Riemann-Hilbert problems arising in the theory of special functions are meromorphic with respect to the relevant parameters.

Theorem 1 allows us to differentiate the functions $Y(\lambda, x)$ and $u(x)$, and the proof of the differential equation (15) becomes relatively easy. Nevertheless, it involves ingredients which are central to the whole modern theory of integrable systems: the *Lax pair formalism* and the *isomonodromy deformation*. Here is a sketch of the proof.

Let $\theta(\lambda) = \frac{4}{3}\lambda^3 + x\lambda$, let σ_3 denote the Pauli matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and put $\Psi(\lambda) = Y(\lambda) e^{-i\theta(\lambda)\sigma_3}$. The diagonal matrix $e^{-i\theta(\lambda)\sigma_3}$ conjugates the jump matrix $G(\lambda)$ into constant matrices:

$$G(\lambda) = e^{-i\theta(\lambda)\sigma_3} S_k e^{i\theta(\lambda)\sigma_3}, \quad \lambda \in \Gamma_k, \tag{17}$$

where S_k is upper (lower) triangular if k is even (odd), has unit diagonal, and has s_k as its nontrivial off-diagonal entry. In terms of the function Ψ, the jump relation becomes

$$\Psi_+(\lambda) = \Psi_-(\lambda) S_k, \quad \lambda \in \Gamma_k, \tag{18}$$

while the normalization condition at $\lambda = \infty$ transforms into the asymptotic condition

$$\Psi(\lambda) = \left(I + O\left(\tfrac{1}{\lambda}\right)\right) e^{-i\theta(\lambda)\sigma_3}. \tag{19}$$

The main point is that the λ- and x-independence of the matrices S_k implies that the "logarithmic derivatives"

$$A(\lambda) := \Psi_\lambda \Psi^{-1} \quad \text{and} \quad U(\lambda) := \Psi_x \Psi^{-1} \tag{20}$$

have no jumps across the rays Γ_k. In addition, the cyclic relation (13), which can be rewritten as the matrix equation

$$S_1 S_2 \dots S_6 = I, \tag{21}$$

implies that the logarithmic derivatives $\Psi_\lambda \Psi^{-1}$ and $\Psi_x \Psi^{-1}$ have no singularities at $\lambda = 0$. Hence the matrix functions $A(\lambda)$ and $U(\lambda)$ are entire functions of λ. In view of (19), we have that $\Psi_\lambda \Psi^{-1} = -4i\lambda^2\sigma_3 + \cdots$ and $\Psi_x \Psi^{-1} = -i\lambda\sigma_3 + \cdots$ as $\lambda \to \infty$. Hence the entire functions $A(\lambda)$ and $U(\lambda)$ are, in fact, polynomials of the second and the first degree, respectively:

$$A(\lambda) = -4i\lambda^2\sigma_3 + \lambda A_1 + A_0$$

and

$$U(\lambda) = -i\lambda\sigma_3 + U_0.$$

The matrix coefficients A_1, A_0, and U_0 can be easily evaluated in terms of the matrix coefficients m_j of the asymptotic series (16). After elementary algebra and a few more technical tricks, the matrix coefficients A_1, A_0, and U_0 can be expressed in terms of a single functional parameter $u \equiv u(x)$ defined according to (14) (note that $u = 2(m_1)_{12}$). In fact, the following equations result:

$$A(\lambda) = -4i\lambda^2\sigma_3 - 4\lambda u\sigma_2 - 2u_x\sigma_1 - (ix + 2iu^2)\sigma_3 \tag{22}$$

and

$$U(\lambda) = -i\lambda\sigma_3 - u\sigma_2, \tag{23}$$

where σ_1 and σ_2 denote the Pauli matrices $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, respectively.

After establishing the polynomial structure of $A(\lambda)$ and $U(\lambda)$, we can reinterpret (20) as saying that the matrix function $\Psi(\lambda) \equiv \Psi(\lambda, x)$ is a solution of the *linear* overdetermined system

$$\begin{cases} \Psi_\lambda = A(\lambda)\Psi, \\ \Psi_x = U(\lambda)\Psi. \end{cases} \tag{24}$$

The compatibility condition $\Psi_{\lambda x} = \Psi_{x\lambda}$ yields the following relation on the coefficient matrices:

$$U_\lambda(\lambda) - A_x(\lambda) = [A(\lambda), U(\lambda)], \quad \text{identically in } \lambda. \tag{25}$$

[13] *Theorem 1 as stated was proved recently by Bolibruch, Kapaev, and this author. It may also properly be viewed as a refinement of earlier work of Fokas and Zhou and of Novokshenov and this author, and apparently it also can be extracted from more general results of Malgrange, Palmer, and Mason and Woodhouse.*

A straightforward calculation shows, in view of (22) and (23), that this matrix identity is equivalent to the scalar differential equation (15) for the functional parameter $u(x)$.

According to the terminology of integrable systems, the linear system (24) and the nonlinear equation (25) are the Lax pair and the zero-curvature (or Lax) representation of the nonlinear ordinary differential equation (15).[14]

The Lax pair (24) puts the nonabelian Airy Riemann-Hilbert problem into the context of the theory of linear ODEs with rational coefficients. Notice, however, that the λ-equation of the system (24) is not Fuchsian. Its only singular point is the *irregular* singular point at $\lambda = \infty$. This means that the fundamental solutions of the equation behave exponentially as $\lambda \to \infty$. In fact, this is exactly the behavior which is indicated by equation (19). Simultaneously, equation (18) manifests the relevant *Stokes phenomenon*, that is, different fundamental solutions with the same asymptotics. In this context, the matrices S_k are the *Stokes multipliers*, and the set $\{S_k\}$ represents a set of *generalized monodromy data* of the first equation in (24). This implies that, similar to the Fuchsian case, the Riemann-Hilbert problem itself can be interpreted as an example of the inverse monodromy problem. In fact, this is the first nontrivial case of the *Riemann-Hilbert-Birkhoff problem*, i.e., the inverse monodromy problem for linear systems allowing irregular singularities.

The x-independence of the matrices S_k, which is responsible for the x-equation in (24), indicates that the zero-curvature equation (25) describes the *isomonodromy deformations* of the λ-equation. Indeed, as was shown in 1980 by Flaschka and Newell, one can derive directly from the zero-curvature equation (25) that the Stokes matrices $S_k \equiv S_k(x, u, u_x)$ of the λ-equation in (24) are the first integrals of motion of the differential equation (15). This and the uniqueness property of Riemann-Hilbert problems (an easy fact) yield the following strengthening of the statement concerning the representation (14) for solutions of differential equation (15).

Proposition 1. *The map defined by equation (14) is a bijection of the algebraic manifold*

$$\{s = (s_1, s_2, s_3) \in \mathbb{C}^3 : s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0\}$$

into the set of solutions of the differential equation (15). In particular, the notation $u(x) \equiv u(x; s)$ for solutions of (15) is justified.

An important corollary of Theorem 1 and Proposition 1 is the following global analytic property of the solutions of equation (15).

Proposition 2. *Every solution of the differential equation (15) is a meromorphic function of the complex variable x. If s is the corresponding monodromy data, then the set of poles of the solution coincides with the set X_s of points where the nonabelian Airy Riemann-Hilbert problem fails to be solvable.*

It is now time to reveal the name of the differential equation (15). It is (a particular case) of the second equation from the Painlevé-Gambier list of ordinary differential equations having the so-called Painlevé property. A second-order differential equation of the form $u_{xx} = F(x, u, u_x)$, where F is meromorphic in x and rational in u and u_x, is said to have the Painlevé property if every solution has a meromorphic continuation to the universal covering of a punctured x-Riemann sphere which is determined by the equation only. This is a statement concerning the global behavior of a general solution, and as such it is often taken as a definition of the very concept of integrability.[15]

At the turn of the last century, Painlevé and Gambier showed that there exist, up to proper transformations of the dependent and independent variables, only fifty equations satisfying the Painlevé property. Moreover, each of these equations can be either integrated by quadrature or reduced to a linear equation or reduced to one of a list of *six* nonlinear equations (for more details see [11]). These six equations, which are called the Painlevé equations, are not integrable in terms of the classical "linear" special functions and classical "nonlinear" special functions (elliptic functions).[16] The solutions of Painlevé equations are called Painlevé functions or Painlevé transcendents. As already indicated, equation (15) is the second equation of Painlevé's list.

---

[14] *Assuming in (17) the abelian reduction, i.e., $s_k = 0$ for $k = 1, 3, 5$, and repeating the same arguments based on the analysis of the logarithmic derivatives $\Psi_\lambda \Psi^{-1}$ and $\Psi_x \Psi^{-1}$, we arrive again at the Lax pair (24), but this time with the upper triangular coefficient matrices $A(\lambda)$ and $U(\lambda)$. The zero-curvature equation (25) in this case is equivalent to the Airy equation. This Lax pair for Airy functions was suggested by Kapaev, Kitaev, and this author in 1988 in a paper where (it seems) the Lax-pair point of view was applied to the classical special functions for the first time. The method of this paper (slightly different from the one presented here) has been further extended by Kitaev in his 2000 paper (*Acta Appl. Math. ***64****) where the development of a unified "isomonodromic" approach to both linear and nonlinear special functions has been essentially completed.*

[15] *Although, as the example of a general linear second-order equation indicates, the Painlevé property does not necessarily yield explicit connection formulae.*

[16] *Strictly speaking, this fact was proved completely rigorously only recently in the works of Umemura and his collaborators in the framework of differential Galois theory.*

It is becoming increasingly evident that Painlevé transcendents should be considered as *new* nonlinear special functions. It is amazing in how many apparently different applications they appear. Here we cannot go further into the modern theory of Painlevé transcendents, so we refer the reader to the monographs [1], [13], [15], and the review paper [12].[17]

Analogously to our discussion of the second Painlevé equation (15), a Riemann-Hilbert formalism can be developed for each of the six Painlevé transcendents, and for all but the first one it can be developed starting from the relevant linear counterpart. In particular, each Painlevé transcendent admits a Riemann-Hilbert representation and an isomonodromy deformation interpretation.[18] These can be used to provide a monodromy data parametrization of the solution manifolds of the Painlevé equations and to prove the analog of Proposition 2 (the Painlevé property) for each of the Painlevé functions. In fact, one can do more.

According to our view of Riemann-Hilbert representations as a nonabelian version of contour integration, one might wonder about the possibility of carrying out a comprehensive global asymptotic analysis of the Painlevé functions, including explicit connection formulae, as x approaches relevant critical points (the "Painlevé-punctures" of the x-Riemann sphere) along different directions in the complex plane. That this can indeed be done was apparently unknown to Painlevé and his contemporaries. To give the reader the flavor of these modern developments, which in our view make the strongest case for the Riemann-Hilbert approach in integrable systems, we present a complete description of the asymptotic behavior of the second Painlevé transcendent $u(x;s)$ as $x \to \pm\infty$ in the case $\bar{s}_3 = -s_1$. This restriction on the monodromy data corresponds to selecting second Painlevé functions that are purely imaginary for real x and hence (by an easy calculation) have no poles on the real line.

Theorem 2.[19] *An arbitrary purely imaginary solution $u(x)$ of the second Painlevé equation (15) has the following oscillatory asymptotic behavior as $x \to -\infty$:*

$$(26)\quad u(x) = i(-x)^{-1/4}\alpha \times \sin\left\{\tfrac{2}{3}(-x)^{3/2} + \tfrac{3}{4}\alpha^2\log(-x) + \varphi\right\} + o((-x)^{-1/4}).$$

Here the constants $\alpha > 0$ and $\varphi \in \mathbb{R}$ (mod 2π) can be any real numbers; they determine the solution $u(x)$ uniquely and hence form a set $\mathbf{a}_- \equiv (\alpha, \varphi)$ of asymptotic parameters at $-\infty$.

Let

$$\Delta(\alpha,\varphi) \equiv \tfrac{3}{2}\alpha^2\log 2 - \tfrac{\pi}{4} - \arg\Gamma\left(i\tfrac{\alpha^2}{2}\right) - \varphi,$$

where $\Gamma(z)$ denotes Euler's gamma function. The behavior of $u(x) \equiv u(x;\alpha,\varphi)$ as $x \to +\infty$ depends on the value of $\Delta(\alpha,\varphi)$. If (generic case)

$$(27)\qquad \Delta(\alpha,\varphi) \neq 0 \pmod{\pi},$$

then the solution $u(x)$ oscillates and tends to $\pm i\sqrt{x/2}$ as $x \to +\infty$; more precisely, one has

$$(28)\quad u(x) = \sigma i\sqrt{\tfrac{x}{2}} + \sigma i(2x)^{-1/4}\rho\cos\left\{\tfrac{2\sqrt{2}}{3}x^{3/2} - \tfrac{3}{2}\rho^2\log x + \theta\right\} + o(x^{-1/4}) \qquad \text{as } x \to +\infty,$$

where $\sigma = \pm$, $\rho > 0$, and $\theta \in \mathbb{R}$ (mod 2π). If instead

$$\Delta(\alpha,\varphi) = 0 \pmod{\pi},$$

then the solution $u(x)$ decreases exponentially as $x \to +\infty$; indeed in this case the asymptotics are the same as for the Airy function $\mathrm{Ai}(x)$; i.e.,

$$(29)\qquad u(x) = \sigma i\frac{\rho}{2\sqrt{\pi}}x^{-1/4}e^{-(2/3)x^{3/2}}(1 + o(1))$$

as $x \to +\infty$, where $\sigma = \pm$ and $\rho > 0$. The set $\mathbf{a}_+$ of asymptotic parameters at $+\infty$ is the triple (ρ,θ,σ) in the generic situation and the pair (ρ,σ) in the special case.

The following explicit connection formulae give the map $\mathbf{a}_- \mapsto \mathbf{a}_+$:

[17] *Painlevé equations have extremely deep relations, due to Clarkson, Dubrovin, Hitchin, Manin, Okamoto, Umemura, and their collaborators, with group theory and algebraic geometry. We do not touch upon these at all, nor upon issues of the explicit particular solutions of Painlevé equations (Gromak, Lukashevich, Tsegel'nik) and the Hamiltonian formalism (Boalch, Flaschka, Harnad, Krichever, Newell, Okamoto). In fact, all these subjects can be also treated in the framework of the Riemann-Hilbert approach, although not all the connections (e.g., with Okamoto's parametrization of the space of initial data) are completely clear at the moment.*

[18] *It also should be mentioned that the isomonodromy interpretation of all six Painlevé equations was first obtained in the classical works of Fuchs, Garnier, and Schlesinger. It was rediscovered and put in the context of the modern theory of integrable systems by Flaschka and Newell, and by Miwa, Jimbo, and Ueno in the early 1980s. The Lax pair (22)–(24) was first obtained by Flaschka and Newell as a result of a self-similar reduction of the Lax pair for the mKdV equation.*

[19] *These results were first obtained in 1986 by Kapaev and this author; some technical gaps that were present in the original proof were filled in by the papers of Deift and Zhou (1995) and Fokas, Kapaev, and this author (1994).*

(30)
$$\rho^2 = \alpha^2 - \frac{1}{\pi}\log\left(2(e^{\pi\alpha^2}-1)^{1/2}|\sin\Delta(\alpha,\varphi)|\right),$$
$$\theta = -\frac{3\pi}{4} - \frac{7}{2}\rho^2\log 2 + \arg\Gamma(i\rho^2)$$
$$+ \arg\left(1+(e^{\pi\alpha^2}-1)e^{2i\Delta(\alpha,\varphi)}\right),$$
$$\sigma = -\mathrm{sign}\,(\sin\Delta(\alpha,\varphi)),$$

if $\Delta(\alpha,\varphi) \neq 0 \pmod{\pi}$*, and*

(31) $$\rho = (e^{\pi\alpha^2}-1)^{1/2}, \quad \sigma = (-1)^n,$$

if $\Delta(\alpha,\varphi) = n\pi$.[20]

Here are some remarks about Theorem 2.

(a) Some parts of the theorem can be obtained without using the Riemann-Hilbert formalism. This is true for the existence, for a given pair (α,φ), of a solution $u(x)$ with the asymptotics (26) (the works of Abdulaev, 1985) or the asymptotics (28) and (29). Since these are *local* statements, they do not reflect the integrability of the second Painlevé equation. The *global* fact that formulae (26) and (28)–(29) describe *all* the possible types of asymptotic behavior of the purely imaginary solutions of the second Painlevé equation (15) as $x \to \pm\infty$ can also be proved, in principle, without appealing to the Riemann-Hilbert representation—this was done in the 1988 and 1992 works of Joshi and Kruskal, but already one has to make use of the integrability of equation (15); indeed, Joshi-Kruskal's constructions essentially exploit the Painlevé property.

Without using the Riemann-Hilbert formalism, it does not seem feasible to obtain the parts of the theorem concerning the bifurcation condition (27) and the connection formulae (30)–(31).

(b) The derivation of the connection formulae (30)–(31) is based on the prior evaluation, *via the asymptotic analysis of the second Painlevé Riemann-Hilbert problem*, of the asymptotic parameters $\mathbf{a}_\pm$ in terms of the monodromy data s_1. The connection formulae (30)–(31) follow by eliminating the *common* parameter s_1 from the equations $\mathbf{a}_\pm = \mathbf{a}_\pm(s_1)$.

(c) There are two major approaches to the asymptotic analysis of the oscillatory Riemann-Hilbert problems appearing in the theory of integrable systems and, in particular, in the theory of Painlevé equations. The first scheme, the *isomonodromy method* developed in the 1980s and 1990s in the works of Andreev, Kapaev, Kitaev, Novokshenov, Suleimanov, and this author, is based on the asymptotic solution of the direct monodromy problem for the corresponding λ-equation and on the interpretation of the monodromy data as first integrals of motion of the Painlevé equations. This approach was inspired by the pioneering paper of Zakharov and Manakov (1976) on the asymptotic analysis of integrable PDEs, and its implementation required a nontrivial development (see, e.g., the works of Kapaev on the first and second Painlevé equations and the 1998 work of Bassom, Clarkson, Law, and McLeod on the second Painlevé equation) of the classical WKB method in the complex domain (see the monograph [13] and the review article [12] for more details; see also the recent works of Bleher and this author on the asymptotics of orthogonal polynomials). The second approach, the nonlinear steepest descent method, was developed, as already mentioned, in the beginning of the 1990s by Deift and Zhou. The Deift-Zhou approach suggests an extremely elegant direct asymptotic analysis of the relevant Riemann-Hilbert problems, so that no prior information about the asymptotic behavior of the solutions is needed. The essence of the nonlinear steepest descent method was briefly outlined in the first section of this survey.

(d) The first connection formulae for specific families of Painlevé transcendents were obtained in 1977 by Ablowitz and Segur (the one-parameter family (29) of the solutions of (15)) and by McCoy, Tracy, and Wu (a one-parameter family of solutions of the third Painlevé equation arising in the 2D Ising model). Ablowitz and Segur used the Zakharov-Manakov formulae and the fact that the second Painlevé equation is a self-similar reduction of the KdV equation. The Ablowitz-Segur connection formulae were rigorously justified in the later work of Clarkson and McLeod and of Suleimanov. The work of McCoy, Tracy, and Wu was actually the first rigorous work on the Painlevé connection formulae. Remarkably, it was done before the discovery of the Riemann-Hilbert formalism for Painlevé equations. There is, however, an important technical point specific to the one-parameter family considered in the McCoy-Tracy-Wu work: it admits a certain Fredholm determinant representation, which, in a sense, is a "shadow" of the Riemann-Hilbert formalism. The "Fredholm determinant branch" of the Riemann-Hilbert approach has been further developed in the recent works of Tracy and Widom devoted to some important special classes of solutions of an integrable higher-order generalization of the third Painlevé equation.

(e) A large number of results concerning the asymptotic descriptions in the full complex domain, including the connection formulae and an explicit evaluation of the distributions of poles near the relevant critical points, have been already obtained for the first five Painlevé equations, mostly in the works of Kapaev, Kitaev, and Novokshenov. The asymptotics and connection formulae for a generic case of the Painlevé VI equation were evaluated via

[20] *It is worth noticing that in the limit of small α the special-case connection formulae (31) become the classical Airy connection formulae.*

the Riemann-Hilbert approach in 1982 by Jimbo. Important special cases, not covered by Jimbo's results, were worked out in the recent works of Doyon, Dubrovin, Guzzetti, and Mazzocco.

A Riemann-Hilbert Problem for the Riemann Zeta-function

We think it is relevant to conclude this article with the following simple observation, which brings in the name of Riemann in one more fundamental way and which the reader might find intriguing.

Starting with the Riemann integral representation for the Riemann zeta-function $\zeta(s)$ (see, e.g., the book of Titchmarsh), we should, according to (3)–(6), arrive at a representation of type (5) for $\zeta(s)$ in terms of the solution $Y(\lambda;s)$ of the Riemann-Hilbert problem posed, with the proper regularization, on the positive real line ($\lambda > 0$) with the jump matrix defined by the equation

$$G(\lambda;s) = \begin{pmatrix} 1 & \pi i \lambda^{\frac{s}{2}-1}\theta_3(0;i\lambda) \\ 0 & 1 \end{pmatrix}, \tag{32}$$

where $\theta_3(z;\tau) = \sum e^{\pi i \tau m^2 + 2\pi i z m}$ is the Jacobi theta-function.[21] Put $\Psi(\lambda) = Y(\lambda)\lambda^{(\frac{s}{4}-\frac{5}{8})\sigma_3}$ and consider the "logarithmic derivatives"

$$\sigma_3 \Psi\left(\frac{1}{\lambda}; 5-s\right)\sigma_3\Psi^{-1}(\lambda;s)$$

and

$$\Psi(\lambda; s+2)\Psi^{-1}(\lambda;s).$$

Due to the well-known symmetry properties of the theta-constant, the Ψ jump matrix is invariant with respect to the transformations $(\lambda, s) \to (1/\lambda,\ 5-s)$ and $s \to s+2$. Analogously to the derivation of the differential Lax pair (24), these properties would lead to the "discrete-discrete" Lax pair

$$\begin{cases} \sigma_3 \Psi\left(\frac{1}{\lambda}; 5-s\right)\sigma_3 = A(s)\Psi(\lambda;s), \\ \Psi(\lambda; s+2) = U(\lambda;s)\Psi(\lambda;s) \end{cases}$$

whose compatibility condition has the form of the cyclic equation

$$\sigma_3 U\left(\frac{1}{\lambda}; 3-s\right)\sigma_3 A(s+2)U(\lambda;s)A^{-1}(s) = I,$$

and it yields, surely, the classical functional equation for $\zeta(s)$. One can now proceed with the process of nonabelianization of the Riemann-Hilbert problem (32) and arrive at a notion of a "nonlinear" Riemann zeta-function.

We are not taking this construction too seriously. At least not yet!

[21] *The regularization mentioned consists in the replacement of $\theta_3(0;i\lambda)$ by $\theta_3(0;i\lambda) - 1$ for $\lambda > 1$ and by $\theta_3(0;i\lambda) - \lambda^{-1/2}$ for $0 < \lambda < 1$.*

References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Math. Soc. Lecture Notes Series, vol. 149, Cambridge University Press, Cambridge, 1991.

[2] D. V. Anosov and A. A. Bolibruch, *The Riemann-Hilbert Problem*, Aspects of Mathematics, E 22, Braunschweig, 1994.

[3] R. Beals, P. A. Deift, and C. Tomei, *Direct and Inverse Scattering on the Line*, Mathematical Surveys and Monographs, vol. 28, Amer. Math. Soc., Providence, RI, 1988.

[4] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer Series in Nonlinear Dynamics, Springer-Verlag, 1994.

[5] P. A. Deift, Integrable operators, in *Differential Operators and Spectral Theory*, Amer. Math. Soc. Transl., Ser. 2, vol. 189, pp. 69–84, 1999.

[6] ____ , *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, Courant Lecture Notes in Mathematics, vol. 3, CIMS, New York, 1999.

[7] ____ , Integrable systems and combinatorial theory, *Notices of the Amer. Math. Soc.* **47** (6) (2000), 631–640.

[8] P. A. Deift, A. R. Its, and X. Zhou, Long-time asymptotics for integrable nonlinear wave equations, in *Important Developments in Soliton Theory* (A. S. Fokas and V. E. Zakharov, eds.), Springer-Verlag, pp. 181–204, 1993.

[9] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, Nonlinear equations of Korteweg–de Vries type, finite-zone linear operators, and Abelian varieties, *Russian Math. Surveys* **31** (1) (1976), 59–146.

[10] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer-Verlag, Berlin, Heidelberg, 1987.

[11] E. L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.

[12] A. R. Its, Connection formulae for the Painlevé transcendents, in *The Stokes Phenomenon and Hilbert's 16th Problem* (B. L. J. Braaksma, G. K. Immink, and M. van der Put, eds.), World Scientific, Singapore, pp. 139–165, 1996.

[13] A. R. Its and V. Yu. Novokshenov, *The Isomonodromic Deformation Method in the Theory of Painlevé Equations*, Lecture Notes in Math., vol. 1191, Springer-Verlag, Heidelberg, 1986.

[14] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, Cambridge, 1993.

[15] D. Levi and P. Winternitz, eds., *Painlevé Transcendents (Their Asymptotics and Physical Applications)*, Plenum Press, New York, 1992.

[16] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons (The Inverse Scattering Method)*, Consultants Bureau, New York, 1984.

[17] M. A. Semenov-Tian-Shansky, Integrable systems and factorization problems, in *Factorization and Integrable Systems*, Operator Theory: Advances and Applications, vol. 141, Birkhäuser-Verlag Basel, Switzerland, 2003.